On higher-power moments of $\Delta(x)$(II)

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Abstract

Let $\Delta(x)$ be the error term of the Dirichlet divisor problem. The asymptotic formula of the integral $\int_1^T \Delta^k(x)dx$ is established for any integer $3 \leq k \leq 9$ by an unified method. Similar results are also established for some other well-known error terms in the analytic number theory .

1 Introduction and main results

1.1 Notations

Throughout this paper, let $d(n)$ denote the Dirichlet divisor function, $r(n)$ denote the number of ways n can be written as $n = x^2 + y^2$ for $x, y \in \mathbb{Z}$, and $a(n)$ denote the Fourier coefficients of a holomorphic cusp form of weight $\kappa = 2n \geq 12$ for the full modular group , $\tilde{a}(n) := a(n)n^{-\kappa/2+1/2}$. For short, we use $d, r, a, \tilde{a}$ denote these functions, respectively. $\zeta(s)$ denotes the Riemann zeta-function.

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Suppose $x > 0, t > 0$. Define

$$\Delta(x) := \sum_{n \le x} d(n) - x \log x - (2\gamma - 1)x, \tag{1.1}$$

$$P(x) := \sum_{n \le x} r(n) - \pi x, \tag{1.2}$$

$$A(x) := \sum_{n \le x} a(n), \tag{1.3}$$

$$E(t) := \int_0^t |\zeta(\frac{1}{2} + iu)|^2 du - t \log(t/2\pi) - (2\gamma - 1)t. \tag{1.4}$$

Suppose $f : \mathbb{N} \to \mathbb{R}$ is any function such that $f(n) \ll n^{\varepsilon}$, $k \ge 2$ is a fixed integer. Define

$$s_{k;l}(f) := \sum_{\sqrt{n_1} + \cdots + \sqrt{n_l} = \sqrt{n_{l+1}} + \cdots + \sqrt{n_k}} \frac{f(n_1) \cdots f(n_k)}{(n_1 \cdots n_k)^{3/4}} \quad (1 \le l < k), \tag{1.5}$$

$$B_k(f) := \sum_{l=1}^{k-1} \binom{k-1}{l} s_{k;l}(f) \cos \frac{\pi(k - 2l)}{4}. \tag{1.6}$$

We shall use $s_{k;l}(f)$ to denote both of the series (1.5) and its value. We will prove the convergence of $s_{k;l}(f)$ in Section 3.

Suppose $A_0 > 2$ is a real number, define

$$\begin{aligned}
K_0 &:= \min\{n \in \mathbb{N} : n \ge A_0, 2|n\}, \\
b(k) &:= 2^{k-2} + \frac{k-6}{4}, \\
\sigma(k, A_0) &:= \begin{cases} 1/4, & \text{if } k - 1 < A_0/2\ , \\ \frac{A_0 - k}{2(A_0 - 2)}, & \text{if } A_0/2 + 1 \le k < A_0, \end{cases} \\
\delta_1(k, A_0) &:= \sigma(k, A_0)/2b(K_0), \\
\delta_2(k, A_0) &:= \frac{\sigma(k, A_0)}{2b(k) + 2\sigma(k, A_0)}.
\end{aligned}$$

$\mathbb{N}$ denotes the set of all natural numbers. ε always denotes a sufficiently small positive constant which may be different at different places. We will use the inequality $d(n) \ll n^{\varepsilon}$ freely. $SC(\Sigma)$ denotes the summation condition of the sum Σ. $\mu(n)$ is the Möbius function.

1.2 Introduction

In this paper we shall study the higher-power moments of $\Delta(x)$, $P(x)$, $A(x)$ and $E(t)$.

We begin with the Dirichlet divisor problem. Dirichlet first proved that $\Delta(x) = O(x^{1/2})$. The exponent 1/2 was improved by many authors. The latest result reads

$$\Delta(x) \ll x^{23/73}(\log x)^{315/146}, \tag{1.7}$$

which can be found in Huxley[6]. It is conjectured that

$$\Delta(x) = O(x^{1/4+\varepsilon}), \tag{1.8}$$

which is supported by the classical mean-square result

$$\int_1^T \Delta^2(x)dx = \frac{(\zeta(3/2))^4}{6\pi^2\zeta(3)}T^{3/2} + O(T\log^5 T) \tag{1.9}$$

proved by Tong[17] and the upper bound estimate

$$\int_1^T |\Delta(x)|^{A_0}dx \ll T^{1+A_0/4+\varepsilon}, \tag{1.10}$$

where $A_0 > 2$ is a fixed real number. The estimate of type (1.10) can be found in Ivić[7, Thm. 13.9] with $A_0 = 35/4$ and Heath-Brown[5] with $A_0 = 28/3$. On the other hand , Voronoi[19] proved that

$$\int_1^T \Delta(x)dx = T/4 + O(T^{3/4}), \tag{1.11}$$

which in conjunction with (1.9) shows that $\Delta(x)$ has a lot of sign change s and cancelations between the positive and negative portions.

Tsang[18] first studied the third- and fourth-power moments of $\Delta(x)$. He proved that (with notations in Section 1.1)

$$\int_1^T \Delta^3(x)dx = \frac{3s_{3;1}(d)}{28\pi^3}T^{7/4} + O(T^{7/4-1/14+\varepsilon}), \tag{1.12}$$

$$\int_1^T \Delta^4(x)dx = \frac{3s_{4;2}(d)}{64\pi^4}T^2 + O(T^{2-1/23+\varepsilon}). \tag{1.13}$$

Heath-Brown[5] proved that for $k = 3, 4, 5, 6, 7, 8, 9$ the limit

$$\lim_{T\to\infty} T^{-1-k/4}\int_1^T \Delta(x)^k dx$$

exists.

In [20] the author improved Tsang's method and proved that

$$\int_1^T \Delta^3(x)dx = \frac{3s_{3;1}(d)}{28\pi^3}T^{7/4} + O(T^{3/2+\varepsilon}), \tag{1.14}$$

$$\int_1^T \Delta^4(x)dx = \frac{3s_{4;2}(d)}{64\pi^4}T^2 + O(T^{2-2/41}), \tag{1.15}$$

$$\int_1^T \Delta^5(x)dx = \frac{5(2s_{5;2}(d) - s_{5;1}(d))}{288\pi^5}T^{9/4} + O(T^{9/4-5/816}). \tag{1.16}$$

But the argument of [20] fails for $k \geq 6$.

1.3 New results on higher-power moments of $\Delta(x)$

In this paper we shall use a different approach to study the higher-power moment s of $\Delta(x)$. This leads to the asymptotic formulas of the integral $\int_1^T \Delta^k(x)dx$ for $3 \leq k \leq 9$. Furthermore, if the estimate (1. 8) is true, then our approach can give the asymptotic formulas of $\int_1^T \Delta^k(x)dx$ for any $k \geq 10$.

Theorem 1. Let $A_0 > 9$ be a real number such that (1.10) holds , then for any integer $3 \leq k < A_0$, we have the asymptotic formula

$$\int_1^T \Delta^k(x)dx = \frac{B_k(d)}{(1+k/4)2^{3k/2-1}\pi^k}T^{1+k/4} + O(T^{1+k/4-\delta_1(k,A_0)+\varepsilon}) \tag{1.17}$$

Remark 1.1. From Ivić's argument[7, Thm 13.9], we know that the value of A_0 for which (1.10) holds depends on the large-value estimate and the upper bound estimate of $\Delta(x)$. If we insert the estimate (1.7) into the argument of Ivić, we get that (1.10) holds with $A_0 = 184/19$. Whence for $k \in \{3, 4, 5, 6, 7, 8, 9\}$, we can get the asymptotic formula (1.17). Moreover, if the estimate $\Delta(x) \ll x^{5/16-\delta}$ holds for some small $\delta > 0$, then the asymptotic formula (1.17) holds for $k = 10$.

Remark 1.2. For $k \geq 10$, Theorem 1 is only an conditional result. However, it tells us that for any $k \geq 10$, the main term in the asymptotic formula of $\int_1^T \Delta^k(x)dx$(if it exists) must have the form stated in (1.17).

Remark 1.3. We can state the following three conjectures about $\Delta(x)$.

Conjecture 1: The estimate (1.8) is true.

Conjecture 2: The estimate (1.10) is true for any $A_0 > 2$.

Conjecture 3: For any fixed $k \geq 3$, there exists a constant $\delta_k > 0$ such that the asymptotic formula

$$\int_1^T \Delta^k(x)dx = \frac{B_k(d)}{(1+k/4)2^{3k/2-1}\pi^k} T^{1+k/4} + O(T^{1+k/4-\delta_k+\varepsilon})$$

holds.

It is well-known that Conjecture 1 and Conjecture 2 are equivalent. From Theorem 1 we know that actually the three Conjectures are equivalent. It is easy to deduce Conjecture 2 from Conjecture 3. To deduce Conjecture 3 from Conjecture 2, we take $A_0 = 2(k-1)$ and $\delta_k = \delta_1(k, 2(k-1))$.

Remark 1.4. From (1.11) we know that the integral $\int_1^T \Delta(x)dx$ have many cancelations from the positive and negative portions of $\Delta(x)$. However, from (1.12) Tsang[18] observed that this is not so for $\int_1^T \Delta^3(x)dx$. From Theorem 1 we know this is also not so for $\int_1^T \Delta^k(x)dx$ $(k = 5, 7, 9)$ since numerical computation tells $B_k(d) > 0$ for $k = 5, 7, 9$. Maybe $B_k(d) > 0$ holds for any odd $k \geq 3$.

The constant $\delta_1(k, A_0)$ is small for k small. If we combine Ivić's argument in the proof of Theorem 1, we can get the following Theorem 2 for $3 \leq k \leq 9$. Note that the results for $k = 3, 4$ are weaker than those of [20]. Theorem 2 for $k = 5$ improved (1.16).

Theorem 2. For $3 \leq k \leq 9$, the asymptotic formula (1.17) holds with $\delta_1(k, A_0)$ replaced by $\delta_2(k, 184/19)$.

Especially for $k = 5, 6, 7, 8, 9$, we have

$$\int_1^T \Delta^5(x)dx = \frac{5(2s_{5;2}(d) - s_{5;1}(d))}{288\pi^5} T^{9/4} + O(T^{9/4-1/64+\varepsilon}), \tag{1.18}$$

$$\int_1^T \Delta^6(x)dx = \frac{5s_{6;3}(d) - 3s_{6;1}(d)}{320\pi^6} T^{5/2} + O(T^{5/2-35/4742+\varepsilon}), \tag{1.19}$$

$$\int_1^T \Delta^7(x)dx = \frac{7(5s_{7;3}(d) - 3s_{7;2}(d) - s_{7;1}(d))}{2816\pi^7} T^{11/4} + O(T^{11/4-17/6312+\varepsilon}), \tag{1.20}$$

$$\int_1^T \Delta^8(x)dx = \frac{7(5s_{8;4}(d) - 4s_{8;2}(d))}{6144\pi^8} T^{3} + O(T^{3-8/9433+\varepsilon}), \tag{1.21}$$

$$\int_1^T \Delta^9(x)dx = \frac{3(3s_{9;1}(d) - 12s_{9;2}(d) - 28s_{9;3}(d) + 42s_{9;4}(d))}{26624\pi^9} T^{13/4} \tag{1.22}$$

$$+O(T^{13/4-13/75216+\varepsilon}).$$

1.4 Higher-power moments of $P(x), A(x)$ and $E(t)$

The method of proving Theorem 1 and Theorem 2 can also be applied to study the higher-power moments of $P(x), A(x)$ and $E(t)$.

The conjectured bound of $P(x)$ is

$$P(x) = O(x^{1/4+\varepsilon}), \tag{1.23}$$

which is supported by

$$\int_2^T P^2(x)dx = (\frac{1}{3\pi^2}\sum_{n=1}^{\infty} r^2(n)n^{-3/2})T^{3/2} + O(T\log^2 T) \tag{1.24}$$

proved by Katai[14]. Tsang[18] also studied the third- and the fourth-power moments of $P(x)$. His results were improved in the author[20]. An asymptotic formula for the fifth-power moment of $P(x)$ was also obtained in [20], which is further improved by the following Theorem 3($k = 5$.)

Theorem 3. Let $A_0 > 9$ be a real number such that the estimate

$$\int_1^T |P(x)|^{A_0}dx \ll T^{1+A_0/4+\varepsilon} \tag{1.25}$$

is true , then for any integer $3 \le k < A_0$, the asymptotic formula

$$\int_1^T P^k(x)dx = \frac{(-1)^k B_k(r)}{(1+k/4)2^{k-1}\pi^k}T^{1+k/4} + O(T^{1+k/4-\delta_1(k,A_0)+\varepsilon}) \tag{1.26}$$

holds.

Especially for $3 \le k \le 9$, the asymptotic formula (1.26) holds with $\delta_1(k, A_0)$ replaced by $\delta_2(k, 184/19)$.

Remark 1.5. Ivić [7, Thm 13.12] proved that the estimate (1.25) holds for $A_0 = 35/4$. If we insert the estimate $P(x) = O(x^{23/73+\varepsilon})$ (see Huxley[6]) into his argument, we find that (1.25) holds for $A_0 = 184/19$.

It is well-known that $A(x)$ has no main term and $A(x) \ll x^{\kappa/2-1/6+\varepsilon}$. From Deligne[4], we have $|\tilde{a}(n)| \le d(n)$.

The conjectured bound of $A(x)$ is $A(x) \ll x^{\kappa/2-1/4+\varepsilon}$. Ivić[9] proved that

$$\int_1^T A^2(x)dx = B_2T^{\kappa+1/2} + O(T^\kappa \log^5 T), \tag{1.27}$$

where

$$B_2 = \frac{1}{4\kappa+2}\sum_{n=1}^{\infty} a^2(n) n^{-\kappa-1/2}.$$

Ivić[9] also proved that the estimate

$$\int_1^T |A(x)|^{A_0} dx \ll T^{1+A_0(2\kappa-1)/4+\varepsilon} \tag{1.28}$$

holds for $A_0 = 8$. Cai [3] studied the third- and fourth-power moments of $A(x)$. His results were improved in the author[20]. In [20] an asymptotic formula for the fifth-pow er moment of $A(x)$ was also obtained, which is further improved by the case $k = 5$ of the following Theorem 4.

Theorem 4. Let $A_0 \geq 8$ be a real number such that (1.28) is true , then for any $3 \leq k < A_0$, the asymptotic formula

$$\int_1^T A^k(x) dx = \frac{B_k(\tilde{a})}{(1+\frac{k(2\kappa-1)}{4})2^{3k/2-1}\pi^k} T^{1+\frac{k(2\kappa-1)}{4}} + O(T^{1+\frac{k(2\kappa-1)}{4}-\delta_1(k,A_0)+\varepsilon}) \tag{1.29}$$

holds.

Especially for $3 \leq k \leq 7$, the asymptotic formula (1.29) holds with $\delta_1(k, A_0)$ replaced by $\delta_2(k, 8)$.

Many results for $E(t)$ parallel to those for $\Delta(x)$ have been obtained; se e Ivić[8] for a survey. The conjectured bound for $E(t)$ is $E(t) \ll t^{1/4+\varepsilon}$, which is supported by

$$\int_2^T E^2(t) dt = \frac{2\zeta^4(3/2)}{3\zeta(3)\sqrt{2\pi}} T^{3/2} + O(T \log^5 T), \tag{1.30}$$

proved by Meurman[15]. It has been proved that (see Huxley[6])

$$E(t) \ll t^{72/227} (\log t)^{629/227}, \quad t > 2. \tag{1.31}$$

Ivić[7, Thm. 15.7] proved that the estimate

$$\int_1^T |E(t)|^{A_0} dt \ll T^{1+A_0/4+\varepsilon} \tag{1.32}$$

holds for $A_0 = 35/4$. Inserting the estimate (1.31) into Ivić's argument, we find that (1.32) is true for $A_0 = 576/61$.

Tsang[18] studied the third- and fourth-power moment of $E(t)$ by using the Atkinson's formula[1]. His results were further improved by Ivić[10] in

a different way. The author[20] obtained new results on the third and the fourth power moments of $E(t)$. An asymptotic formula for the fifth power moment of $E(t)$ was also obtained in [20], which is further improved by the case $k=5$ of the following Theorem 5..

Theorem 5. Let $A_0 > 9$ be a real number such that the estimates (1.10) and

(1.32) hold , then for any $3 \le k < A_0$, we have the asymptotic formula

$$\int_1^T E^k(t)dt = \frac{B_k(d)}{(1+k/4)2^{3k/4-1}\pi^{k/4}} T^{1+k/4} + O(T^{1+k/4-\delta_1(k,A_0)+\varepsilon}). \quad (1.33)$$

Especially for $3 \le k \le 9$, the asymptotic formula (1.33) holds with $\delta_1(k, A_0)$ replaced by $\delta_2(k, 576/61)$.

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2 Some Preliminary Lemmas

We need the following Lemmas.

Lemma 2.1. The square-roots of squarefree numbers are linearly independent over the integers.

Proof. This is a classical result of Besicovitch[2]. □

Lemma 2.2. Suppose $k \ge 3$, $(i_1, \cdots, i_{k-1}) \in \{0,1\}^{k-1}$ such that

$$\sqrt{n_1} + (-1)^{i_1}\sqrt{n_2} + (-1)^{i_2}\sqrt{n_3} + \cdots + (-1)^{i_{k-1}}\sqrt{n_k} \ne 0.$$

Then we have

$$|\sqrt{n_1} + (-1)^{i_1}\sqrt{n_2} + (-1)^{i_2}\sqrt{n_3} + \cdots + (-1)^{i_{k-1}}\sqrt{n_k}| \gg \max(n_1, \cdots, n_k)^{-(2^{k-2}-2^{-1})}.$$

Proof. The cases $k = 3, 4$ are Lemma 1 and Lemma 2 of Tsang[18], respectively. The proof for the general case is the same as the proof of Lemma 1 of [18]. We note that Heath-Brown[5] stated a similar result for k even. □

Lemma 2.3. Suppose $A, B \in \mathbb{R}, A \ne 0$, then

$$\int_T^{2T} \cos(A\sqrt{t} + B)dt \ll T^{1/2}|A|^{-1}.$$

Lemma 2.4. Suppose $k \geq 3$, $(i_1, \cdots, i_{k-1}) \in \{0,1\}^{k-1}$, $(i_1, \cdots, i_{k-1}) \neq (0, \cdots, 0)$, $1 < N_1, N_2, \cdots, N_k$, $0 < \Delta \ll E^{1/2}$, $E = \max(N_1, N_2, \cdots, N_k)$. Let

$$\mathcal{A} = \mathcal{A}(N_1, N_2, \cdots, N_k; i_1, \cdots, i_{k-1}; \Delta)$$

denote the number of solutions of the inequality

$$|\sqrt{n_1} + (-1)^{i_1}\sqrt{n_2} + (-1)^{i_2}\sqrt{n_3} + \cdots + (-1)^{i_{k-1}}\sqrt{n_k}| < \Delta \tag{2.1}$$

with $N_j < n_j \leq 2N_j, 1 \leq j \leq k$. Then

$$\mathcal{A} \ll \Delta E^{-1/2} N_1 N_2 \cdots N_k + E^{-1} N_1 N_2 \cdots N_k.$$

Proof. Without loss of generality, suppose $E = N_k$. If $(n_1, \cdots, n_k)$ satisfies (2. 1), then

$$\sqrt{n_1} + (-1)^{i_1}\sqrt{n_2} + (-1)^{i_2}\sqrt{n_3} + \cdots + (-1)^{i_{k-2}}\sqrt{n_{k-1}} = (-1)^{i_{k-1}+1}\sqrt{n_k} + \theta\Delta$$

for some $|\theta| < 1$. Whence we get

$$\left(\sqrt{n_1} + (-1)^{i_1}\sqrt{n_2} + (-1)^{i_2}\sqrt{n_3} + \cdots + (-1)^{i_{k-2}}\sqrt{n_{k-1}}\right)^2 = n_k + O(\Delta N_k^{1/2}).$$

Hence for fixed $(n_1, \cdots, n_{k-1})$, the number of n_k is $\ll 1 + \Delta N_k^{1/2}$ and thus

$$\mathcal{A} \ll \Delta N_k^{1/2} N_1 N_2 \cdots N_{k-1} + N_1 N_2 \cdots N_{k-1}.$$

□

3 On the series $s_{k;l}(d)$

In this section we shall study the series $s_{k;l}(d)$. Suppose $y > 1$ is a large parameter, and define

$$s_{k;l}(d; y) := \sum_{\substack{\sqrt{n_1} + \cdots + \sqrt{n_l} = \sqrt{n_{l+1}} + \cdots + \sqrt{n_k} \\ n_1, \cdots, n_k \leq y}} \frac{d(n_1) \cdots d(n_k)}{(n_1 \cdots n_k)^{3/4}}, 1 \leq l < k.$$

We shall prove the following Lemma 3.1.

Lemma 3.1. We have

$$|s_{k;l}(d) - s_{k;l}(d; y)| \ll y^{-1/2+\varepsilon}, 1 \leq l < k.$$

Remark. Lemma 3.1 is still true if the divisor function d is replaced b y any function $f : \mathbb{N} \to \mathbb{R}$ with $f(n) \ll n^{\varepsilon}$.

Proof. We shall prove Lemma 3.1 by induction in k. The case $k = 2$ is easy. The case $k = 3$ is contained already in Page 70 of Tsang[18], later we always suppose $k \geq 4$. Since $s_{k;l}(d) = s_{k;k-l}(d)$, we suppose $l \leq k/2$.

By the symmetry, we get

$$|s_{k;l}(d) - s_{k;l}(d;y)| \ll \sum_{\substack{\sqrt{n_1}+\cdots+\sqrt{n_l}=\sqrt{n_{l+1}}+\cdots+\sqrt{n_k} \\ n_1>y}} \frac{d(n_1)\cdots d(n_k)}{(n_1\cdots n_k)^{3/4}} \tag{3.1}$$

$$\ll U_1(d;y) + U_2(d;y),$$

say, where

$$U_1(d;y) := \sum_{j=l+1}^{k} \sum_{\substack{\sqrt{n_1}+\cdots+\sqrt{n_l}=\sqrt{n_{l+1}}+\cdots+\sqrt{n_k} \\ n_1=n_j>y}} \frac{d(n_1)\cdots d(n_k)}{(n_1\cdots n_k)^{3/4}},$$

$$U_2(d;y) := \sum_{\substack{\sqrt{n_1}+\cdots+\sqrt{n_l}=\sqrt{n_{l+1}}+\cdots+\sqrt{n_k} \\ n_1>y, n_1\neq n_j, l+1\leq j\leq k}} \frac{d(n_1)\cdots d(n_k)}{(n_1\cdots n_k)^{3/4}}.$$

If $l = 1$, then obviously $U_1(d;y) = 0$. If $l > 1$, then by induction we get

$$U_1(d;y) \ll \sum_{n>y} \frac{d^2(n)}{n^{3/2}} s_{k-2;l-1}(d) \ll y^{-1/2+\varepsilon}. \tag{3.2}$$

Now we estimate $U_2(d;y)$. Let $I = \{1,\cdots,l\}$, $J = \{l+1,\cdots,k\}$. Suppose $(n_1,\cdots,n_k) \in \mathbb{N}^k$ such that

$$(*): \quad \sqrt{n_1}+\cdots+\sqrt{n_l} = \sqrt{n_{l+1}}+\cdots+\sqrt{n_k}, n_1 \neq n_j, l+1 \leq j \leq k.$$

Then there exist two sets $I_0 \subset I$, $J_0 \subset J$ which satisfy the following properties:

(1). $1 \in I_0$;

(2). $\sum_{i\in I_0} \sqrt{n_i} = \sum_{j\in J_0} \sqrt{n_j}$;

(3). For any real subset $I_0^{'} \subset I_0, J_0^{'} \subset J_0$, we have

$$\sum_{i\in I_0^{'}} \sqrt{n_i} \neq \sum_{j\in J_0^{'}} \sqrt{n_j}.$$

If $(I_0, J_0) = (I, J)$, then we say $(n_1,\cdots,n_k)$ is a primitive $(k,l)-$poi nt. Let $\mathbb{N}_{k;l}$ denote the set of all points in $\mathbb{N}^k$ which satisfy (*) and $\mathbb{N}_{k;l}^*$ the set of all

primitive $(k,l)-$points, respectively. Let $\mathcal{G}_{k;l}$ denote the set of all possible pairs (I_0, J_0) when $(n_1, \cdots, n_k)$ runs through $\mathbb{N}_{k;l}$. Note that if $l=1$, then $\mathcal{G}_{k;l} = \{(I,J)\}$.

Suppose $(I_0, J_0) \in \mathcal{G}_{k;l}$. Let $l_1 = \#I_0, l_2 = l - l_1$, $k_1 = \#I_0 + \#J_0, k_2 = k - k_1$. From (*), we know that $k_1 \geq 3$. Define

$$R_1^{(I_0,J_0)}(d;y) := \sum_{\substack{\sqrt{n_1}+\cdots+\sqrt{n_{l_1}}=\sqrt{n_{l_1+1}}+\cdots+\sqrt{n_{k_1}} \\ n_1>y,(n_1,\cdots,n_{k_1})\in\mathbb{N}^*_{k_1;l_1}}} \frac{d(n_1)\cdots d(n_{k_1})}{(n_1\cdots n_{k_1})^{3/4}}.$$

If $(I_0, J_0) \neq (I,J)$, then $l_1 < l, k_1 < k$ and we define

$$R_2^{(I_0,J_0)}(d) := \sum_{\sqrt{m_1}+\cdots+\sqrt{m_{l_2}}=\sqrt{m_{l_2+1}}+\cdots+\sqrt{m_{k_2}}} \frac{d(m_1)\cdots d(m_{k_2})}{(m_1\cdots n_{k_2})^{3/4}}.$$

By the induction assumption, $R_2^{(I_0,J_0)}(d) \ll 1$.

If$(n_1, \cdots, n_{k_1}) \in \mathbb{N}^*_{k_1;l_1}$, then by Lemma 2.1 we have

$$n_j = s_j^2 h, s_1 + \cdots + s_{l_1} = s_{l_1+1} + \cdots + s_{k_1}, \mu(h) \neq 0.$$

$n_1 > y$ implies that there exists at least one $n_j(l_1+1 \leq j \leq j_1)$ such t hat $n_j \gg y$. We suppose $n_{k_1} \gg y$. So we have

$$\begin{aligned}
R_1^{(I_0,J_0)}(d;y) \quad & \ll \sum_h \sum_{\substack{s_1+\cdots+s_{l_1}=s_{l_1+1}+\cdots+s_{k_1} \\ s_1^2 h>y, s_{k_1}^2 h\gg y}} \frac{d(s_1^2 h)\cdots d(s_{k_1}^2 h)}{h^{3k_1/4}(s_1\cdots s_{k_1})^{3/2}} \\
& \ll \sum_h \sum_{\substack{s_1+\cdots+s_{l_1}=s_{l_1+1}+\cdots+s_{k_1} \\ s_1^2 h>y, s_{k_1}^2 h\gg y}} \frac{d^2(s_1)\cdots d^2(s_{k_1}) d^{k_1}(h)}{h^{3k_1/4}(s_1\cdots s_{k_1})^{3/2}} \\
& \ll \sum_h \frac{d^{k_1}(h)}{h^{3k_1/4}} \sum_{s_1>(y/h)^{1/2}} \frac{d^2(s_1)}{s_1^{3/2}} \sum_{s_{k_1}\gg(y/h)^{1/2}} \frac{d^2(s_{k_1})}{s_{k_1}^{3/2}} \\
& \ll \sum_h \frac{d^{k_1}(h)}{h^{3k_1/4}} (\frac{y}{h})^{-1/2+\varepsilon} \ll y^{-1/2+\varepsilon}
\end{aligned}$$

if we notice $k_1 \geq 3$.

If $\mathcal{G}_{k;l} = (I,J)$, we have

$$U_2(d;y) \ll R_1^{(I,J)}(d;y) \ll y^{-1/2+\varepsilon}. \tag{3.3}$$

If $\mathcal{G}_{k;l} \neq (I, J)$, we have

$$U_2(d;y) \ll R_1^{(I,J)}(d;y) + \sum_{\substack{(I_0,J_0)\in\mathcal{G}_{k;l}\\(I_0,J_0)\neq(I,J)}} R_1^{(I_0,J_0)}(d;y)R_2^{(I_0,J_0)}(d) \ll y^{-1/2+\varepsilon}. \tag{3.4}$$

Now Lemma 3.1 follows from (3.1)-(3.4). □

4 Proofs of Theorem 1 and Theorem 2

Suppose $T \geq 10$ is a real number. It suffices for us to evaluate the integral $\int_T^{2T} \Delta^k(x)dx$. Suppose y is a parameter such that $T^\varepsilon < y \leq T^{1/3}$. For any $T \leq x \leq 2T$, define

$$\mathcal{R}_1 = \mathcal{R}_1(x,y) := (\sqrt{2}\pi)^{-1}x^{1/4}\sum_{n\leq y}\frac{d(n)}{n^{3/4}}\cos(4\pi\sqrt{xn}-\frac{\pi}{4}),$$
$$\mathcal{R}_2 = \mathcal{R}_2(x,y) := \Delta(x) - \mathcal{R}_1.$$

We shall show that the higher-power moment of $\mathcal{R}_2$ is small and hence the integral $\int_T^{2T} \Delta^k(x)dx$ can be well approximated by $\int_T^{2T} \mathcal{R}_1^k dx$, which is easy to evaluate.

4.1 Evaluation of the integral $\int_T^{2T} \mathcal{R}_1^h dx$

Suppose $h \geq 3$ is any fixed integer. By the elementary formula

$$\cos a_1 \cdots \cos a_h = \frac{1}{2^{h-1}} \sum_{(i_1,\cdots,i_{h-1})\in\{0,1\}^{h-1}} \cos(a_1+(-1)^{i_1}a_2+(-1)^{i_2}a_3+\cdots+(-1)^{i_{h-1}}a_h),$$

we have

$$\begin{aligned}
\mathcal{R}_1^h \quad &= (\sqrt{2}\pi)^{-h}x^{\frac{h}{4}}\sum_{n_1\leq y}\cdots\sum_{n_h\leq y}\frac{d(n_1)\cdots d(n_h)}{(n_1\cdots n_h)^{3/4}}\prod_{j=1}^{h}\cos(4\pi\sqrt{n_j x}-\pi/4)\\
&= \frac{x^{\frac{h}{4}}}{(\sqrt{2}\pi)^h 2^{h-1}}\sum_{(i_1,\cdots,i_{h-1})\in\{0,1\}^{h-1}}\sum_{n_1\leq y}\cdots\sum_{n_h\leq y}\frac{d(n_1)\cdots d(n_h)}{(n_1\cdots n_h)^{3/4}}\\
&\qquad \times\cos(4\pi\sqrt{x}\alpha(n_1,\cdots,n_h;i_1,\cdots,i_{h-1})-\frac{\pi}{4}\beta(i_1,\cdots,i_{h-1})),
\end{aligned}$$

where

$$\alpha(n_1,\cdots,n_h;i_1,\cdots,i_{h-1})$$
$$:=\sqrt{n_1}+(-1)^{i_1}\sqrt{n_2}+(-1)^{i_2}\sqrt{n_3}+\cdots+(-1)^{i_{h-1}}\sqrt{n_h},$$
$$\beta(i_1,\cdots,i_{h-1}):=1+(-1)^{i_1}+(-1)^{i_2}+\cdots+(-1)^{i_{h-1}}.$$

Thus we can write

$$\mathcal{R}_1^h=\frac{1}{(\sqrt{2}\pi)^h 2^{h-1}}(S_1(x)+S_2(x)), \tag{4.1}$$

where

$$\begin{aligned}
S_1(x):\quad &=x^{h/4}\sum_{(i_1,\cdots,i_{h-1})\in\{0,1\}^{h-1}}\cos(-\frac{\pi\beta}{4})\sum_{\substack{n_j\le y,1\le j\le h\\ \alpha=0}}\frac{d(n_1)\cdots d(n_h)}{(n_1\cdots n_h)^{3/4}},\\
S_2(x):\quad &=x^{h/4}\sum_{(i_1,\cdots,i_{h-1})\in\{0,1\}^{h-1}}\sum_{\substack{n_j\le y,1\le j\le h\\ \alpha\ne 0}}\frac{d(n_1)\cdots d(n_h)}{(n_1\cdots n_h)^{3/4}}\cos(4\pi\alpha\sqrt{x}-\frac{\pi\beta}{4}),\\
\alpha:\quad &=\alpha(n_1,\cdots,n_h;i_1,\cdots,i_{h-1}),\\
\beta:\quad &=\beta(i_1,\cdots,i_{h-1}).
\end{aligned}$$

First consider the contribution of $S_1(x)$. We have

$$\int_T^{2T}S_1(x)dx=\sum_{(i_1,\cdots,i_{h-1})\in\{0,1\}^{h-1}}\cos(-\frac{\pi\beta}{4})\sum_{\substack{n_j\le y,1\le j\le h\\ \alpha=0}}\frac{d(n_1)\cdots d(n_h)}{(n_1\cdots n_h)^{3/4}}\int_T^{2T}x^{\frac{h}{4}}dx. \tag{4.2}$$

It is easily seen that if $\alpha=0$, then $1\in\{i_1,\cdots,i_{h-1}\}$. Let $l=i_1+\cdots+i_{h-1}$, then we have

$$\sum_{\substack{n_j\le y,1\le j\le h\\ \alpha=0}}\frac{d(n_1)\cdots d(n_h)}{(n_1\cdots n_h)^{3/4}}=s_{h;l}(d;y),$$

where $s_{h;l}(d;y)$ was defined in last section. By Lemma 3.1 we get

$$\int_T^{2T}S_1(x)dx=B_h^*(d)\int_T^{2T}x^{\frac{h}{4}}dx+O(T^{1+h/4+\varepsilon}y^{-1/2}), \tag{4.3}$$

where

$$B_h^*(d):=\sum_{(i_1,\cdots,i_{h-1})\in\{0,1\}^{h-1}}\cos(-\frac{\pi\beta}{4})\sum_{\substack{(n_1,\cdots,n_h)\in\mathbb{N}^h\\ \alpha=0}}\frac{d(n_1)\cdots d(n_h)}{(n_1\cdots n_h)^{3/4}}.$$

For any $(i_1, \cdots, i_{h-1}) \in \{0,1\}^{h-1} \setminus \{(0, \cdots, 0)\}$, let

$$\begin{aligned} S(d; i_1, \cdots, i_{h-1}) : \quad &= \sum_{\substack{(n_1,\cdots,n_h)\in\mathbb{N}^h \\ \alpha=0}} \frac{d(n_1)\cdots d(n_h)}{(n_1\cdots n_h)^{3/4}}, \\ l(i_1, \cdots, i_{h-1}) : \quad &= i_1 + \cdots + i_{h-1}. \end{aligned}$$

It is easily seen that if $l(i_1, \cdots, i_{h-1}) = l(i'_1, \cdots, i'_{h-1})$ or $l(i_1, \cdots, i_{h-1}) + l(i'_1, \cdots, i'_{h-1}) = h$, then

$$S(d; i_1, \cdots, i_{h-1}) = S(d; i'_1, \cdots, i'_{h-1}) = s_{h;l(i_1,\cdots,i_{h-1})}(d).$$

From $(-1)^i = 1 - 2i (i = 0, 1)$ we also have

$$\beta(i_1, \cdots, i_{h-1}) = h - 2l(i_1, \cdots, i_{h-1}).$$

So we get

$$B_h^*(d) = \sum_{l=1}^{h-1} \sum_{l(i_1,\cdots,i_{h-1})=l} \cos(-\frac{\pi\beta}{4}) S(d; i_1, \cdots, i_{h-1}) \tag{4.4}$$

$$\begin{aligned} &= \sum_{l=1}^{h-1} s_{h;l}(d) \cos\frac{\pi(h-2l)}{4} \sum_{l(i_1,\cdots,i_{h-1})=l} 1 \\ &= \sum_{l=1}^{h-1} \binom{h-1}{l} s_{h;l}(d) \cos\frac{\pi(h-2l)}{4} = B_h(d). \end{aligned}$$

Now we consider the contribution of $S_2(x)$. By Lemma 2.3 we get

$$\int_T^{2T} S_2(x)dx \ll T^{1/2+h/4} \sum_{(i_1,\cdots,i_{h-1})\in\{0,1\}^{h-1}} \sum_{\substack{n_j\le y, 1\le j\le h \\ \alpha\ne 0}} \frac{d(n_1)\cdots d(n_h)}{(n_1\cdots n_h)^{3/4}|\alpha|}. \tag{4.5}$$

It suffices for us to estimate the sum

$$\Sigma(y; i_1, \cdots, i_{h-1}) = \sum_{\substack{n_j\le y, 1\le j\le h \\ \alpha\ne 0}} \frac{d(n_1)\cdots d(n_h)}{(n_1\cdots n_h)^{3/4}|\alpha|}$$

for fixed $(i_1, \cdots, i_{h-1}) \in \{0,1\}^{h-1}$.

If $(i_1, \cdots, i_{h-1}) = (0, \cdots, 0)$, then

$$\begin{aligned}
&\Sigma(y; 0, \cdots, 0) \ll \sum_{n_j \le y, 1 \le j \le h} \frac{d(n_1) \cdots d(n_h)}{(n_1 \cdots n_h)^{3/4}(\sqrt{n_1} + \cdots + \sqrt{n_h})} \\
&\ll \sum_{n_j \le y, 1 \le j \le h} \frac{d(n_1) \cdots d(n_h)}{(n_1 \cdots n_h)^{3/4+1/2h}} \\
&\ll y^{(h-2)/4} \log^h y,
\end{aligned}$$

where we used the estimates

$$\sum_{n \le u} \ll u \log u, \quad x_1 + \cdots + x_h \gg (x_1 \cdots x_k)^{1/h}.$$

For $(i_1, \cdots, i_{h-1}) \ne (0, \cdots, 0)$, by a splitting argument we get that there exist a group of numbers $1 < N_1, N_2, \cdots, N_h < y$ such that

$$\Sigma(y; i_1, \cdots, i_{h-1}) \ll \Sigma_1^* \log^h y,$$

where

$$\Sigma_1^* = \sum_{\substack{N_j < n_j \le 2N_j, 1 \le j \le h \\ \alpha \ne 0}} \frac{d(n_1) \cdots d(n_h)}{(n_1 \cdots n_h)^{3/4} |\alpha|}.$$

Without loss of generality, we suppose $N_1 \le N_2 \le \cdots \le N_h \le y$.

By Lemma 2.2 we have $|\alpha| \gg N_h^{-(2^{h-2}-2^{-1})}$. Then by a splitting argument and Lemma 2.4 we get for some $N_h^{-(2^{h-2}-2^{-1})} \ll \Delta < y^{1/2}$ such that

$$\begin{aligned}
\Sigma_1^* \quad &\ll \frac{y^\varepsilon}{(N_1 \cdots N_h)^{3/4} \Delta} \mathcal{A}(N_1, \cdots, N_h; i_1, \cdots, i_{h-1}; \Delta) \\
&\ll \frac{y^\varepsilon}{(N_1 \cdots N_h)^{3/4} \Delta} (\Delta N_h^{1/2} N_1 \cdots N_{h-1} + N_1 \cdots N_{h-1}) \\
&\ll y^\varepsilon \left(\frac{(N_1 \cdots N_{h-1})^{1/4}}{N_h^{1/4}} + \frac{(N_1 \cdots N_{h-1})^{1/4}}{N_h^{3/4} \Delta}\right) \\
&\ll y^\varepsilon (N_h^{(h-2)/4} + N_h^{b(h)}) \ll y^{b(h)+\varepsilon},
\end{aligned}$$

where $b(h)$ was defined in Section 1.1. Thus we get

$$\int_T^{2T} S_2(x) dx \ll T^{1/2+h/4+\varepsilon} y^{b(h)}. \tag{4.6}$$

Hence from (4.1)-(4.6) we get

Lemma 4.1. For any fixed $h \geq 3$, we have

$$\int_T^{2T} \mathcal{R}_1^h dx = \frac{B_h(d)}{(\sqrt{2}\pi)^h 2^{h-1}} \int_T^{2T} x^{h/4} dx + O(T^{1+h/4+\varepsilon} y^{-1/2} + T^{1/2+h/4+\varepsilon} y^{b(h)}). \tag{4.7}$$

4.2 Higher-power moments of $\mathcal{R}_2$

We first study the mean-square of $\mathcal{R}_2$. We begin with the truncated Voronoi's formula[9, equation (2.25)]

$$\Delta(x) = (\pi\sqrt{2})^{-1} x^{\frac{1}{4}} \sum_{n \leq N} \frac{d(n)}{n^{3/4}} \cos(4\pi\sqrt{nx} - \pi/4) + O(x^{1/2+\varepsilon} N^{-1/2}), \tag{4.8}$$

where $1 < N \ll x$. Taking $N = T$, we get

$$\begin{aligned} \mathcal{R}_2 \quad &= (\pi\sqrt{2})^{-1} x^{\frac{1}{4}} \sum_{y < n \leq T} \frac{d(n)}{n^{3/4}} \cos(4\pi\sqrt{nx} - \pi/4) + O(T^\varepsilon) \\ &\ll |x^{\frac{1}{4}} \sum_{y < n \leq T} \frac{d(n)}{n^{3/4}} e(2\sqrt{nx})| + T^\varepsilon, \end{aligned}$$

which implies

$$\int_T^{2T} \mathcal{R}_2^2 dx \ll T^{1+\varepsilon} + \int_T^{2T} |x^{\frac{1}{4}} \sum_{y < n \leq T} \frac{d(n)}{n^{3/4}} e(2\sqrt{nx})|^2 dt \tag{4.9}$$

$$\begin{aligned} &\ll T^{1+\varepsilon} + T^{3/2} \sum_{y < n \leq T} \frac{d^2(n)}{n^{3/2}} + T \sum_{y < m < n \leq T} \frac{d(n)d(m)}{(mn)^{3/4}(\sqrt{n} - \sqrt{m})} \\ &\ll T^{1+\varepsilon} + \frac{T^{3/2}\log^3 T}{y^{1/2}} \ll \frac{T^{3/2}\log^3 T}{y^{1/2}}, \end{aligned}$$

where we used the estimates

$$\sum_{n \leq u} d^2(n) \ll u \log^3 u,$$

$$\sum_{y < m < n \leq T} \frac{d(n)d(m)}{(mn)^{3/4}(\sqrt{n} - \sqrt{m})} \ll T^\varepsilon.$$

Now suppose y satisfies $y^{2b(K_0)} \le T$. Hence from Lemma 4.1 we get that

$$\int_T^{2T} |\mathcal{R}_1|^{K_0} dx \ll T^{1+K_0/4+\varepsilon},$$

which implies

$$\int_T^{2T} |\mathcal{R}_1|^{A_0} dx \ll T^{1+A_0/4+\varepsilon} \tag{4.10}$$

since $A_0 \le K_0$. From (1.10) and (4.10) we get

$$\int_T^{2T} |\mathcal{R}_2|^{A_0} dx \ll \int_T^{2T} (|\Delta(x)|^{A_0} + |\mathcal{R}_1|^{A_0}) dx \ll T^{1+A_0/4+\varepsilon}. \tag{4.11}$$

For any $2 < A < A_0$, from (4.9), (4.11) and Hölder's inequality we get that

$$\int_T^{2T} |\mathcal{R}_2|^A dx = \int_T^{2T} |\mathcal{R}_2|^{\frac{2(A_0-A)}{A_0-2}+\frac{A_0(A-2)}{A_0-2}} dx \tag{4.12}$$

$$\ll \left(\int_T^{2T} \mathcal{R}_2^2 dx\right)^{\frac{A_0-A}{A_0-2}} \left(\int_T^{2T} |\mathcal{R}_2|^{A_0} dx\right)^{\frac{A-2}{A_0-2}} \ll T^{1+\frac{A}{4}+\varepsilon} y^{-\frac{A_0-A}{2(A_0-2)}}.$$

Namely, we have the following Lemma 4.2.

Lemma 4.2. Suppose $T^\varepsilon \le y \le T^{1/2b(K_0)}$, $2 < A < A_0$, then

$$\int_T^{2T} |\mathcal{R}_2|^A dx \ll T^{1+\frac{A}{4}+\varepsilon} y^{-\frac{A_0-A}{2(A_0-2)}}. \tag{4.13}$$

4.3 Proof of Theorem 1.

Suppose $3 \le k \le K(A_0)$ and $T^\varepsilon \le y \le T^{1/2b(K_0)}$. By the elementary formula $(a+b)^k - a^k \ll |a^{k-1}b| + |b|^k$, we get

$$\int_T^{2T} \Delta^k(x) dx = \int_T^{2T} \mathcal{R}_1^k dx + O\Big(\int_T^{2T} |\mathcal{R}_1^{k-1}\mathcal{R}_2| dx\Big) + O\Big(\int_T^{2T} |\mathcal{R}_2|^k dx\Big). \tag{4.14}$$

If $k-1 < A_0/2$, then from (4.9), (4.10) and Cauchy's inequality we get

$$\int_T^{2T} |\mathcal{R}_1^{k-1}\mathcal{R}_2| dx \ll \left(\int_T^{2T} |\mathcal{R}_1|^{2(k-1)} dx\right)^{1/2} \left(\int_T^{2T} |\mathcal{R}_2|^2 dx\right)^{1/2} \ll T^{1+k/4+\varepsilon} y^{-1/4}.$$

If $k-1 \geq A_0/2$, then from (4.10), Lemma 4.2 and Hölder's inequality we get

$$\begin{aligned}\int_T^{2T} |\mathcal{R}_1^{k-1}\mathcal{R}_2|dx \quad &\ll \left(\int_T^{2T} |\mathcal{R}_1|^{A_0}dx\right)^{\frac{k-1}{A_0}} \left(\int_T^{2T} |\mathcal{R}_2|^{\frac{A_0}{A_0-k+1}}dx\right)^{\frac{A_0-k+1}{A_0}} \\ &\ll T^{1+k/4+\varepsilon}y^{-(A_0-k)/2(A_0-2)}.\end{aligned}$$

Thus we have

$$\int_T^{2T} |\mathcal{R}_1^{k-1}\mathcal{R}_2|dx + \int_T^{2T} |\mathcal{R}_2|^k dx \ll T^{1+k/4+\varepsilon}y^{-\sigma(k,A_0)}, \tag{4.15}$$

where $\sigma(k, A_0)$ was defined in Section 1.1.

From (4.14) and (4.15) we get

$$\int_T^{2T} \Delta^k(x)dx = \int_T^{2T} \mathcal{R}_1^k dx + O(T^{1+k/4+\varepsilon}y^{-\sigma(k,A_0)}). \tag{4.16}$$

Now take $y = T^{1/2b(K_0)}$. From Lemma 4.1 and (4.16) we get

$$\begin{aligned}\int_T^{2T} \Delta^k(x)dx &= \frac{B_k(d)}{(\sqrt{2}\pi)^k 2^{k-1}} \int_T^{2T} x^{k/4}dx + O(T^{1+k/4-\sigma(k,A_0)/2b(K_0)+\varepsilon}) \\ &= \frac{B_k(d)}{(\sqrt{2}\pi)^k 2^{k-1}} \int_T^{2T} x^{k/4}dx + O(T^{1+k/4-\delta_1(k,A_0)+\varepsilon}).\end{aligned} \tag{4.17}$$

Theorem 1 follows from (4.17) immediately.

4.4 Proof of Theorem 2

Suppose $T^\varepsilon \leq y \leq T^{1/3}$. By the truncated Voronoi's formula (4.8), we have

$$\mathcal{R}_2 = (\sqrt{2}\pi)^{-1}x^{1/4} \sum_{y<n\leq N} \frac{d(n)}{n^{3/4}} \cos(4\pi\sqrt{nx} - \pi/4) + O(x^{1/2+\varepsilon}N^{-1/2}),$$

where $y < N \ll T$. Using Ivić's large-value technique directly to $\mathcal{R}_2$ without modifications , we get that the estimate

$$\int_T^{2T} |\mathcal{R}_2|^{A_0}dx \ll T^{1+A_0/4+\varepsilon} \tag{4.18}$$

holds with $A_0 = 184/19, T^\varepsilon \le y \le T^{1/3}$. We omit the details since it is completely the same as that of Ivić. Combining (1.10) we get that

$$\int_T^{2T} |\mathcal{R}_1|^{A_0} dx \ll T^{1+A_0/4+\varepsilon} \tag{4.19}$$

holds with $A_0 = 184/19, T^\varepsilon \le y \le T^{1/3}$.

By the same argument as in last subsection , we get that for $T^\varepsilon \le y \le T^{1/3}$,

$$\int_T^{2T} \Delta^k(x)dx = \int_T^{2T} \mathcal{R}_1^k dx + O(T^{1+k/4+\varepsilon} y^{-\sigma(k,184/19)}). \tag{4.20}$$

Take $y = T^{1/(2b(k)+2\sigma(k,184/19))}$. From Lemma 4.1 again we get

$$\begin{aligned}\int_T^{2T} \Delta^k(x)dx &= \frac{B_k(d)}{(\sqrt{2}\pi)^k 2^{k-1}} \int_T^{2T} x^{k/4} dx + O(T^{1+k/4-\frac{\sigma(k,184/19)}{2b(k)+2\sigma(k,184/19)}+\varepsilon}) \\ &= \frac{B_k(d)}{(\sqrt{2}\pi)^k 2^{k-1}} \int_T^{2T} x^{k/4} dx + O(T^{1+k/4-\delta_2(k,184/19)+\varepsilon}).\end{aligned} \tag{4.21}$$

And Theorem 2 follows.

5 Proofs of other Theorems

$P(x)$ has the following truncated Voronoi's formula

$$P(x) = -\frac{1}{\pi} \sum_{n\le N} r(n) n^{-3/4} x^{1/4} \cos(4\pi\sqrt{nx} + \pi/4) + O(x^{1/2+\varepsilon} N^{-1/2}) \tag{5.1}$$

for $1 \le N \ll x$, which follows from Lemma 3 of Müller[16]. $A(x)$ has the following truncated Voronoi's formula

$$\begin{aligned}A(x) = \frac{1}{\pi\sqrt{2}} x^{\kappa/2-1/4} \sum_{n\le N} a(n) n^{-\kappa/2-1/4} \cos(4\pi\sqrt{nx} - \pi/4) \\ +O(x^{\kappa/2+\varepsilon} N^{-1/2})\end{aligned} \tag{5.2}$$

for $1 \le N \ll x$, which is a special case of Theorem 1.1 of Jutila[13]. So by the same way as in last section, we get Theorem 3 and Theorem 4 .

Now we prove Theorem 5. We shall follow Ivić[10]. Define

$$\Delta^*(x) := \frac{1}{2} \sum_{n\le 4x} (-1)^n d(n) - x(\log x + 2\gamma - 1), x > 0.$$

Jutila[12] proved that

$$\int_0^T (E(t) - 2\pi\Delta^*(\frac{t}{2\pi}))^2 dt \ll T^{4/3}\log^3 T, \tag{5.3}$$

which means that $E(t)$ is well approximated by $2\pi\Delta^*(\frac{t}{2\pi})$ at least in the mean square sense.

Suppose $A_0 > 9$ is a real number such that both of (1.10) and (1.32) hold. Since (see Jutila[11])

$$\Delta^*(x) = -\Delta(x) + 2\Delta(2x) - \frac{1}{2}\Delta(4x),$$

from (1.10) we get

$$\int_0^T |\Delta^*(t)|^{A_0} dt \ll T^{1+A_0/4+\varepsilon}. \tag{5.4}$$

Then from (1.32), (5.3), (5.4) and Hölder's inequality we get for any $3 \leq k < A_0$ that

$$\int_0^T E^k(t)dt - (2\pi)^{k+1}\int_0^{\frac{T}{2\pi}} (\Delta^*(t))^k dt \tag{5.5}$$

$$\begin{aligned} &= \int_0^T \left(E^k(t) - (2\pi\Delta^*(\frac{t}{2\pi}))^k\right) dt \\ &\ll \int_0^T |E(t) - 2\pi\Delta^*(\frac{t}{2\pi})| \left(|E(t)|^{k-1} + |\Delta^*(\frac{t}{2\pi})|^{k-1}\right) dt \\ &\ll T^{1+k/4-\sigma(k,A_0)/3+\varepsilon}, \end{aligned}$$

where $\sigma(k, A_0)$ was defined in Section 1.1.

From (5.5) the problem is reduced to evaluating the integral $\int_0^T (\Delta^*(t))^k dt$. For $1 \ll N \ll x$, we have[10, equation (7)]

$$\Delta^*(x) = \frac{1}{\pi\sqrt{2}}\sum_{n\leq N}(-1)^n d(n) n^{-3/4} x^{1/4}\cos(4\pi\sqrt{nx}-\pi/4) + O(x^{1/2+\varepsilon}N^{-1/2}), \tag{5.6}$$

which is similar to (4.8). Let $d^*(n) = (-1)^n d(n)$. Then by the same way as i n the proof of Theorem 1, we get that the asymptotic formula

$$\int_1^T (\Delta^*(t))^k dt = \frac{B_k(d^*)}{(1+k/4)2^{3k/2-1}\pi^k} T^{1+k/4} + O(T^{1+k/4-\delta_1(k,A_0)+\varepsilon}) \tag{5.7}$$

holds for any $3 \leq k < A_0$.

We shall use the following Lemma 5.1.

Lemma 5.1 Suppose $1 \leq l < k$ are fixed integers, $(n_1, \cdots, n_k) \in \mathbb{N}^k$. If

$$\sqrt{n_1} + \cdots + \sqrt{n_l} = \sqrt{n_{l+1}} + \cdots + \sqrt{n_k}$$

holds , then $2|(n_1 + \cdots + n_k)$.

Proof. For any $n \in \mathbb{N}$, let $h(n)$ denote the squarefree part of n. Let $\mathcal{S} = \{h(n_1), \cdots, h(n_k)\} \bigcap \mathbb{N}$ and $s = \#\mathcal{S}$. For convenience, write

$$\mathcal{S} = \{h_1, \cdots, h_s\}, I = \{1, \cdots, l\}, J = \{l+1, \cdots, k\}.$$

From Lemma 2.1 we can write $I = \bigcup_{e=1}^s I_e, J = \bigcup_{e=1}^s J_e$ such that for each $1 \leq e \leq s$, we have

$$\sum_{i \in I_e} \sqrt{n_i} = \sum_{j \in J_e} \sqrt{n_j}$$

and that all $n_i (i \in I_e)$ and $n_j (j \in J_e)$ have the same squarefree part h_e. Namely we have $(1 \leq e \leq s)$

$$n_i = m_i^2 h_e (i \in I_e), n_j = m_j^2 h_e (i \in I_e), \sum_{i \in I_e} m_i = \sum_{j \in J_e} m_j.$$

Thus we get

$$\begin{aligned} n_1 + \cdots + n_k &= \sum_{e=1}^s \Big(\sum_{i \in I_e} n_i + \sum_{j \in J_e} n_j\Big) \\ &= \sum_{e=1}^s \Big(\sum_{i \in I_e} m_i^2 h_e + \sum_{j \in J_e} m_j^2 h_e\Big) \equiv \sum_{e=1}^s \Big(\sum_{i \in I_e} m_i + \sum_{j \in J_e} m_j\Big) h_e \\ &= 2 \sum_{e=1}^s h_e \sum_{i \in I_e} m_i \equiv 0 (mod \ \ 2), \end{aligned}$$

where we used the simple congruence $n^2 \equiv n (mod \ \ 2)$. $\square$

From Lemma 5.1 we get for any $1 \leq l < k$ that

$$\begin{aligned} s_{k;l}(d^*) &= \sum_{\sqrt{n_1} + \cdots + \sqrt{n_l} = \sqrt{n_{l+1}} + \cdots + \sqrt{n_k}} (-1)^{n_1 + \cdots + n_k} \frac{d(n_1) \cdots d(n_k)}{(n_1 \cdots n_k)^{3/4}} \\ &= \sum_{\sqrt{n_1} + \cdots + \sqrt{n_l} = \sqrt{n_{l+1}} + \cdots + \sqrt{n_k}} \frac{d(n_1) \cdots d(n_k)}{(n_1 \cdots n_k)^{3/4}} \\ &= s_{k;l}(d). \end{aligned}$$

Whence we get

$$B_k(d^*) = B_k(d). \tag{5.8}$$

From (5.5), (5.7) and (5.8) we get (1.33).

Similar to Theorem 2, we can prove that the asymptotic formula

$$\int_1^T (\Delta^*(t))^k dt = \frac{B_k(d)}{(1+k/4)2^{3k/2-1}\pi^k} T^{1+k/4} + O(T^{1+k/4-\delta_2(k,576/61)+\varepsilon}) \tag{5.9}$$

holds for any $3 \le k \le 9$, which combined with (5.5) yields the second part of Theorem 3.

Note added in proof: Recently M. N. Huxley (Exponential sums and Lattice points III, Proc. London Math. Soc., Vol.**87**(3)(2003), 591-609) proved

$$\Delta(x) \ll x^{131/416} (\log x)^{26947/8320},$$

which implies that the formula (1.10) holds for $A_0 = 262/27$. The exponent $\delta_2(k, 184/19)$ in Theorem 2 then can be improved to $\delta_2(k, 262/27)$ for $k = 6, 7, 8, 9$. The author deeply thanks Professor A. Schinzel for informing me M. N. Huxley's new result.

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